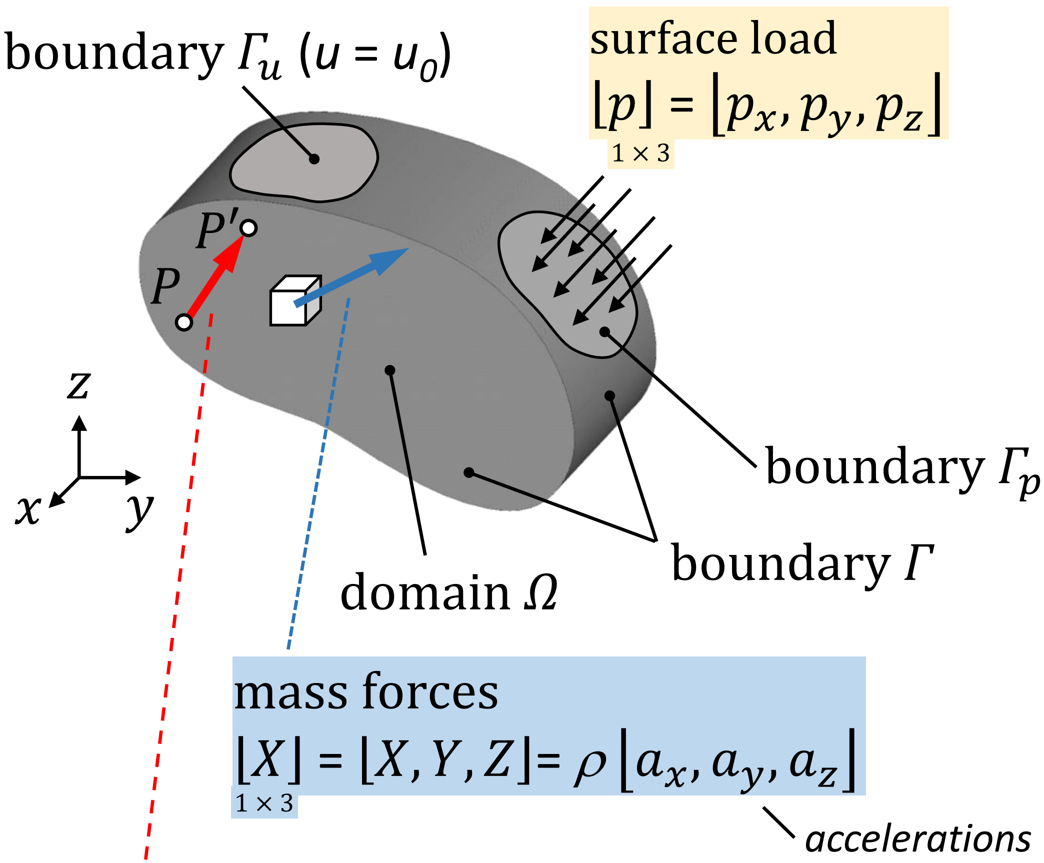
Finite element method (FEM1)

Lecture 2A. The boundary value problem of solid mechanics in the FEM approach

03.2026

Boundary value problem of solid body mechanics



UNKNOWN FUNCTION

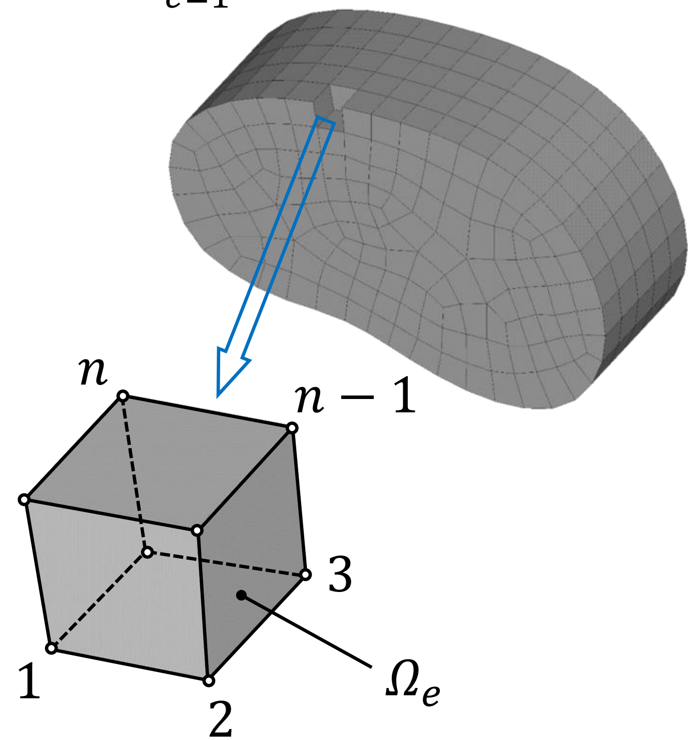
Displacement vector $\underset{3\times 1}{\{u\}} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$

FE model

NOE – no. of FEs

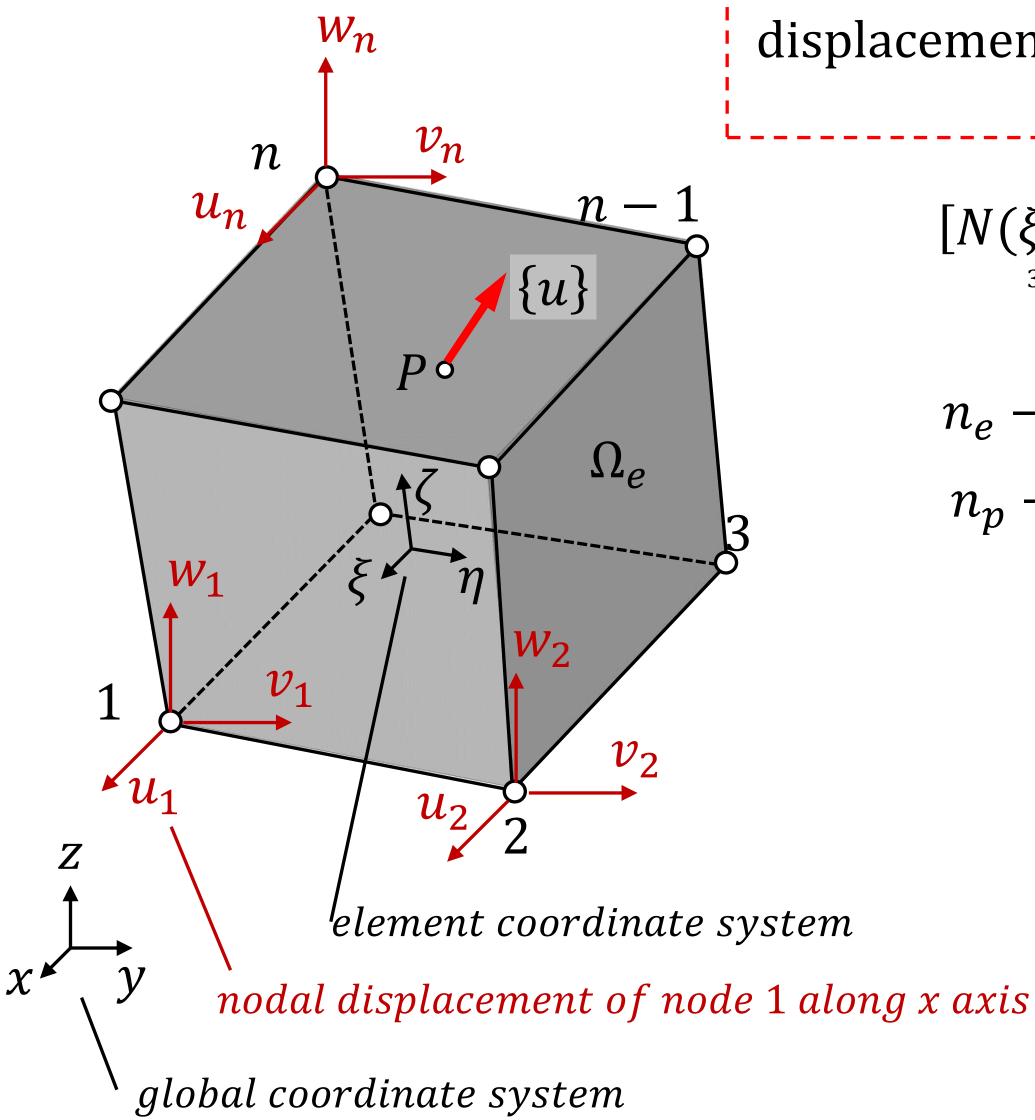
NON – no. of nodes

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \text{ and } \Omega_i \underset{i\neq j}{\cap} \Omega_j = 0$$



Finite element with n - nodes

Nodal approximation inside the finite element with n - nodes



displacement vector $\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$

$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]}$ – matrix of shape funtions

$$n_e = n \cdot n_p$$

n_e – no. of degrees of freedom in FE

n_p – no. of degrees of freedom per node

$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

– local vector of nodal parameters

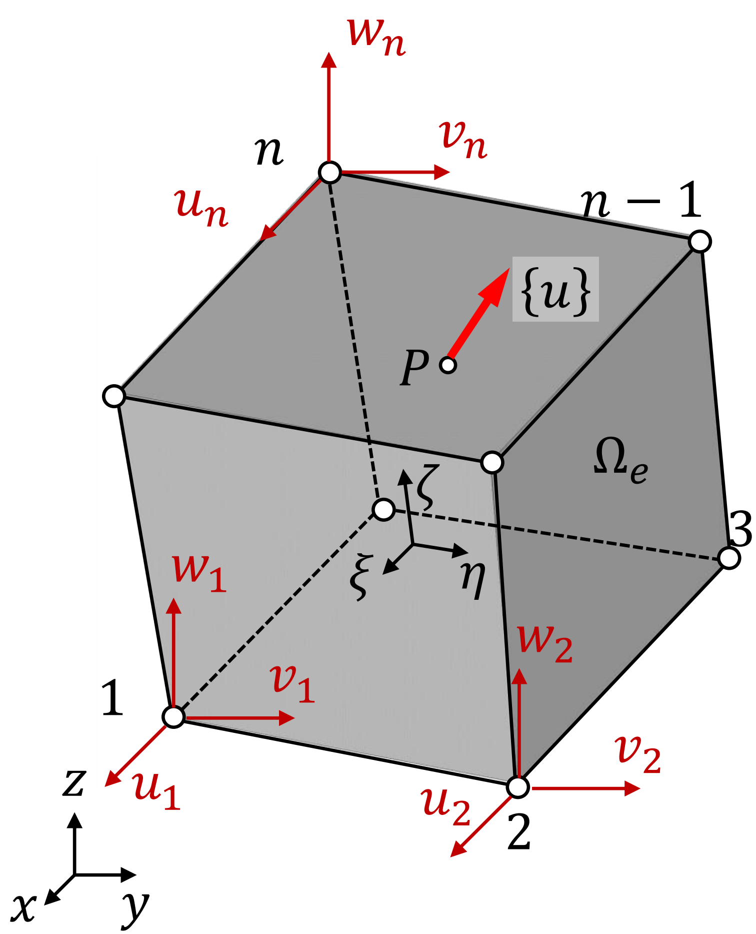
Matrix of shape functions

Shape function matrix:

$$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix}$$

Vector of element nodal parameters:

$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

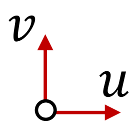
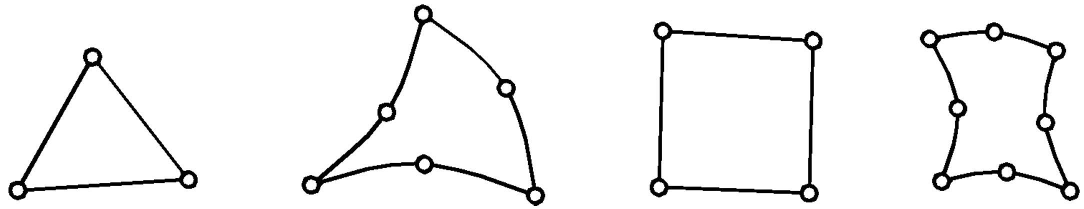
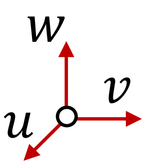
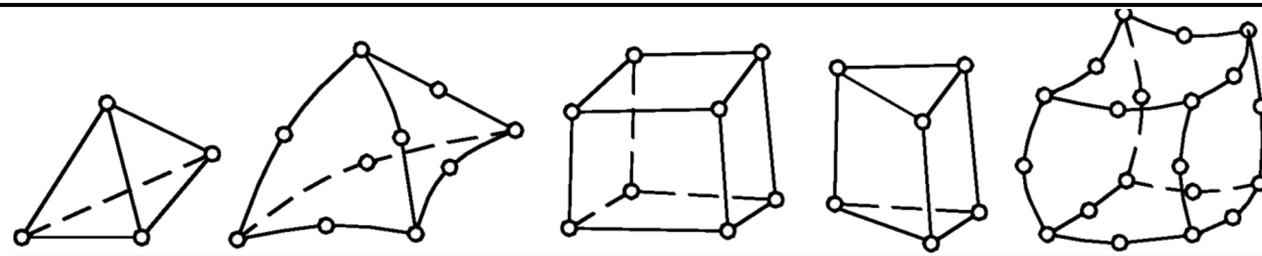
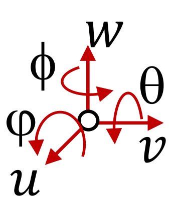


Nodal approximation:

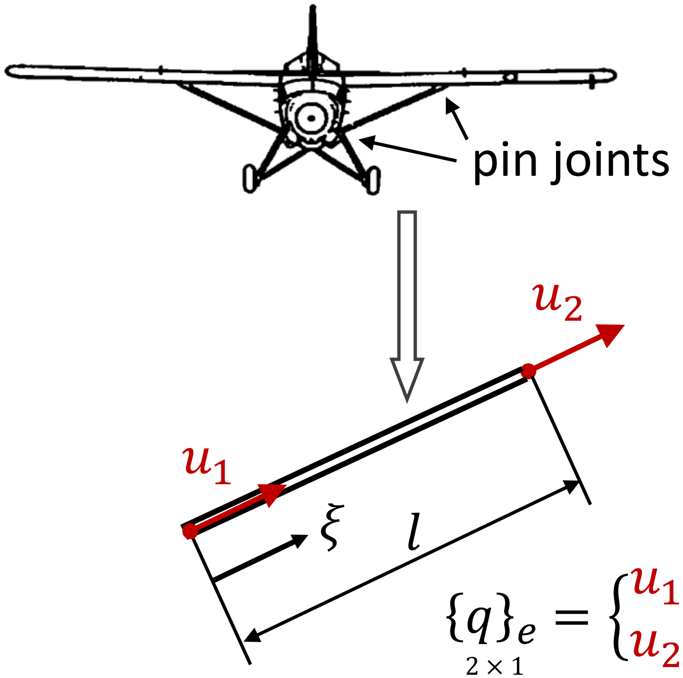
$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$

$$u = N_1 \cdot u_1 + N_2 \cdot u_2 + \dots + N_n \cdot u_n$$
$$v = N_1 \cdot v_1 + N_2 \cdot v_2 + \dots + N_n \cdot v_n$$
$$w = N_1 \cdot w_1 + N_2 \cdot w_2 + \dots + N_n \cdot w_n$$

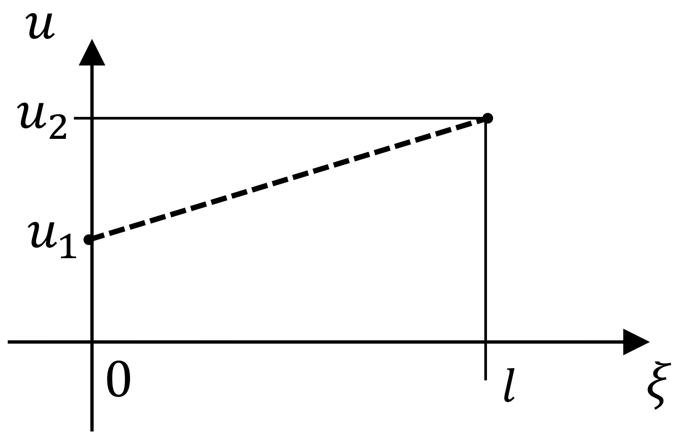
Examples of finite elements

Type	n_e − number of degrees of freedom in FE
rods	 2, 4, 6; 6
2D 	 6; 12; 8; 16
3D 	 12; 30; 24; 18; 60
shell 	 24; 48

Example 1: shape functions for a finite element representing a strut



$$\{q\}_{\underset{2\times 1}{e}} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



linear function:

$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1$$

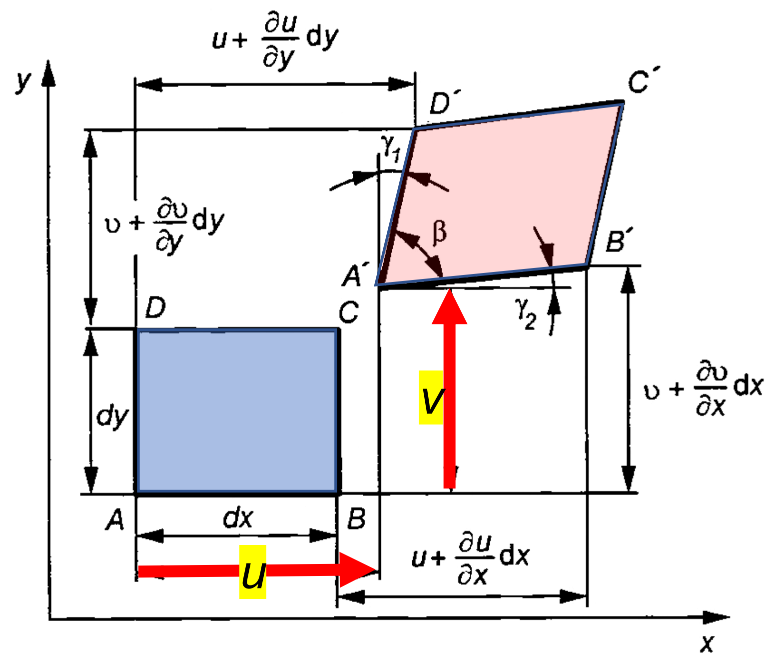
$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1 = \frac{u_2}{l}\xi - \frac{u_1}{l}\xi + u_1 = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 =$$

$$= N_1(\xi) \cdot u_1 + N_2(\xi) \cdot u_2 = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e = \underset{1\times 2}{\lfloor N(\xi) \rfloor} \underset{2\times 1}{\{q\}_e}$$

shape functions: $N_1(\xi) = 1 - \frac{\xi}{l}$; $N_2(\xi) = \frac{\xi}{l}$

Strain components

normal strains:



$$\varepsilon_x = \frac{(A'B')_x - AB}{AB} = \frac{(\mathrm{dx} + u + \frac{\partial u}{\partial x}\,\mathrm{dx} - u) - \mathrm{dx}}{\mathrm{dx}} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

shear strains:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \gamma_1 + \gamma_2$$

$$\gamma_1 \cong \tan\gamma_1 = \frac{(A'D')_x}{(A'D')_y} = \frac{u + \frac{\partial u}{\partial y}\,\mathrm{dy} - u}{\mathrm{dy} + v + \frac{\partial v}{\partial y}\,\mathrm{dy} - v} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{1 + \varepsilon_y} = \frac{\partial u}{\partial y}$$

$$\gamma_2 \cong \frac{\partial v}{\partial x} \quad \rightarrow \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

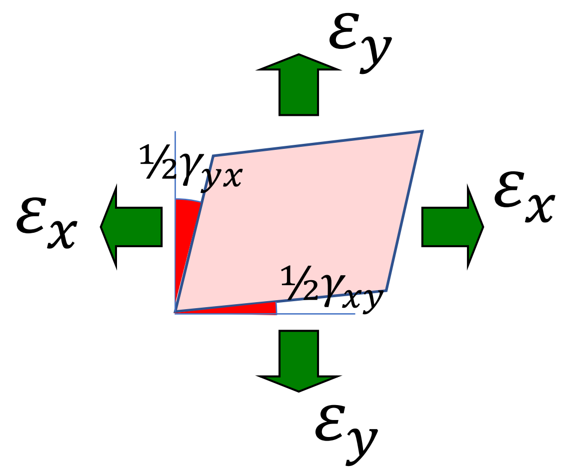
small deformations : $\varepsilon_y \ll 1$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad ; \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad ; \quad \gamma_{ij} = \gamma_{ji}$$

Strain tensor. Vector of strain components

strain tensor:

$$\underset{3\times 3}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$$



vector of strain components:

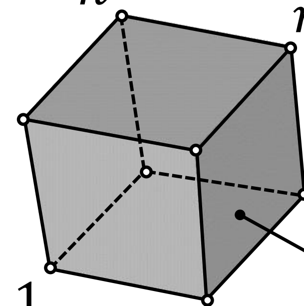
$$\underset{6\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} \quad ; \quad \underset{1\times 6}{\lfloor \varepsilon \rfloor} = \underset{1\times 3}{\lfloor u \rfloor}\underset{3\times 6}{[R]^T}$$

↑ *gradient matrix*

Strain – displacement matrix of a finite element

nodal approximation in a finite element:

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$



vector of strain components in a finite element:

$$\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$$

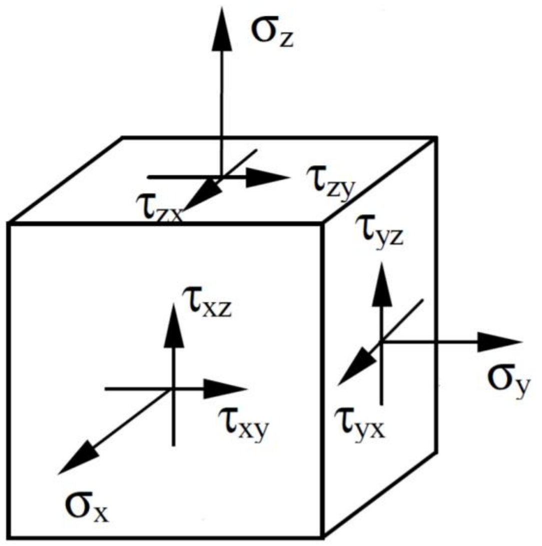
$$\underset{1\times 6}{\lfloor\varepsilon\rfloor} = \underset{1\times n_e}{\lfloor q\rfloor_e}\underset{n_e\times 6}{[B]^T}$$

$$\underset{6\times n_e}{[B]} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]} \quad - \textit{strain–displacement matrix}$$

Stress components

normal stresses: $\sigma_x \;;\; \sigma_y \;;\; \sigma_z$

positive value - tension, negative value - compression



shear stress components: $\tau_{xy} \;;\; \tau_{yz} \;;\; \tau_{zx} \;;\; \tau_{ij} = \tau_{ji}$

equivalent stresses:

Von Mises stress:

$$\sigma_{EQV} = \sqrt{\frac{1}{2}\left(\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2\right) + 3(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2)}$$

Tresca stress: $\sigma_{INT} = \sigma_1 - \sigma_3 = 2\tau_{max}$

the first principal stress (σ_1); *the third principal stress* (σ_3); *maximum shear stress* (τ_{max})

Stress tensor. Vector of stress components

stress tensor:

$$\underset{3\times 3}{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

in (x, y, z) coordinate system — *in the principal coordinate system*

vector of stress components:

$$\underset{6\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Constitutive matrix

linear isotropic material (Hooke's law):

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\,\underset{6\times 1}{\{\varepsilon\}}$$

constitutive matrix:

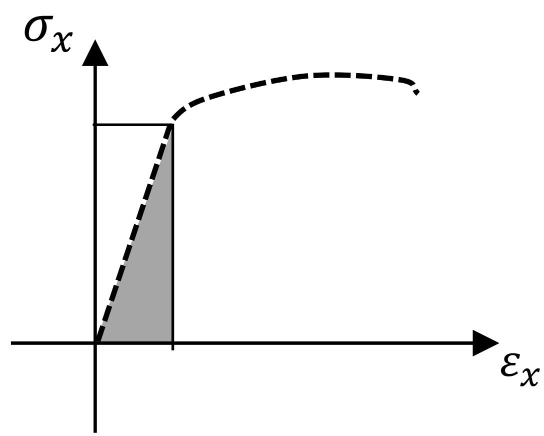
$$\underset{6\times 6}{[D]} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix}$$

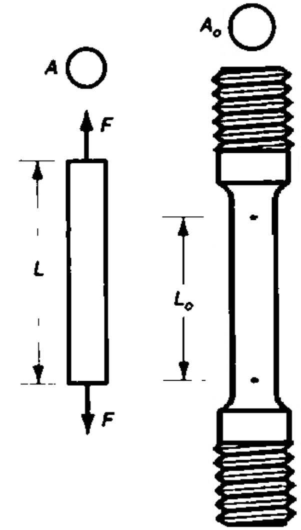
E – Young's modulus,
v – Poisson's ratio

Example 2: uniaxial tensile test

$$\sigma_x = \frac{F}{A_0} \quad ; \quad \varepsilon_x = \frac{L-L_0}{L_0} \quad ; \quad \varepsilon_y = \varepsilon_z = \varepsilon_T$$

elastic strain Energy: $U = \frac{1}{2}\sigma_x\,\varepsilon_x\,A_0\,L_0$





$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\;\underset{6\times 1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} \sigma_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_T \\ \varepsilon_T \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

2nd equation:

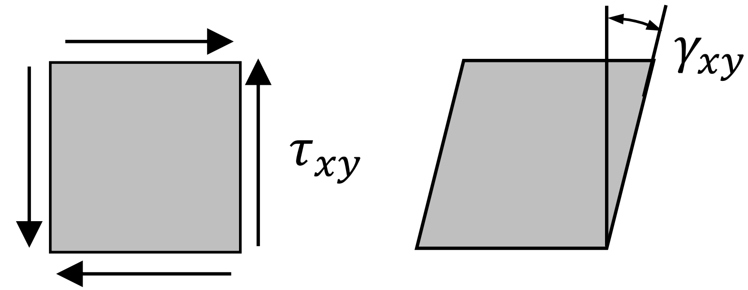
$$0 = \frac{E}{(1+\nu)(1-2\nu)}(v\varepsilon_x + (1-v)\varepsilon_T + v\varepsilon_T) \;\rightarrow\; \boxed{\varepsilon_T = -v\varepsilon_x}$$

1st equation:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}((1-v)\varepsilon_x + v\varepsilon_T + v\varepsilon_T) = \frac{E}{(1-\nu-2v^2)}((1-v)\varepsilon_x - v^2\varepsilon_x - v^2\varepsilon_x) \;\rightarrow\; \boxed{\sigma_x = E\,\varepsilon_x}$$

Example 3: pure shear

τ_{xy} ; γ_{xy}



$$\underset{6\times1}{\{\sigma\}} = \underset{6\times6}{[D]}\ \underset{6\times1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

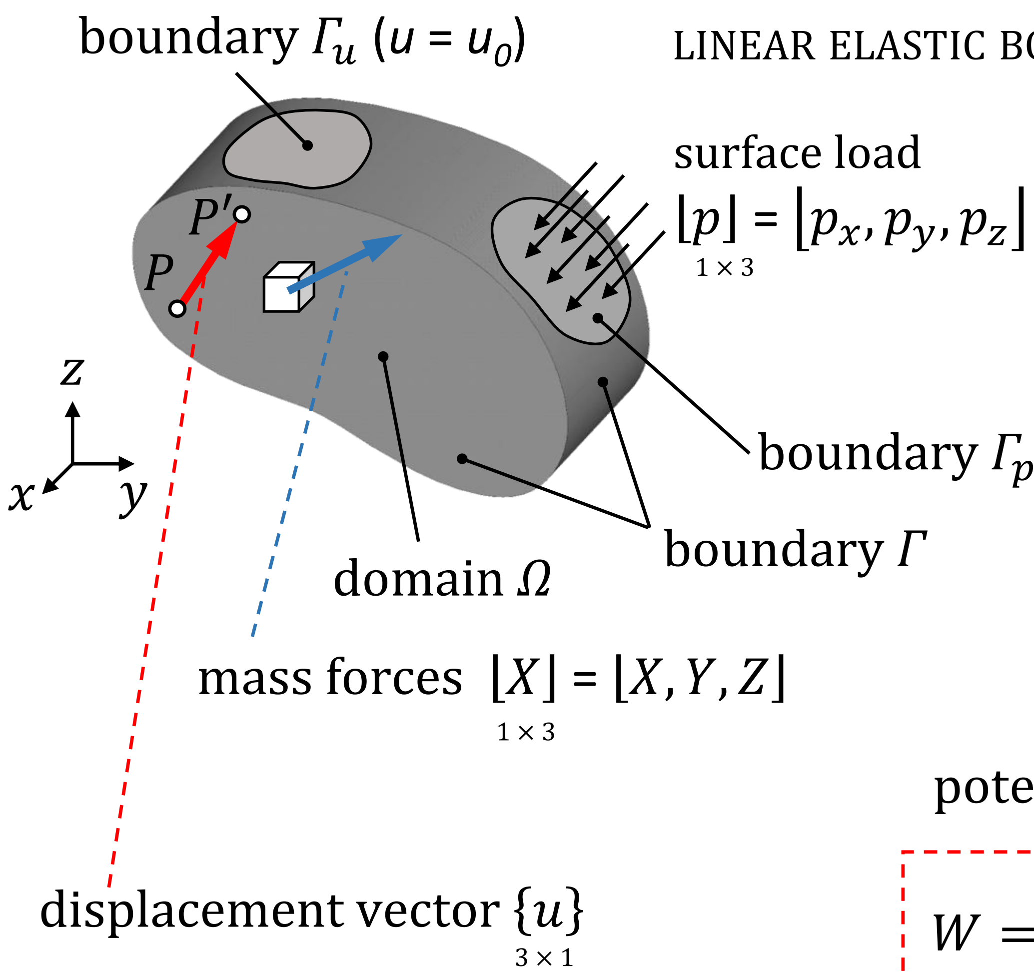
4th equation:

$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)(0.5-\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)}\ \gamma_{xy} \rightarrow$$

$$\tau_{xy} = G\,\gamma_{xy}$$

$G = \frac{E}{2(1+\nu)}$ – *Kirchof's modulus (shear modulus)*

Elastic strain energy. Potential energy of loading



elastic strain energy:

$$U = \frac{1}{2}\int_{\Omega} \lfloor \varepsilon \rfloor_{1\times 6} \{\sigma\}_{6\times 1} \, d\Omega$$

potential energy of loading:

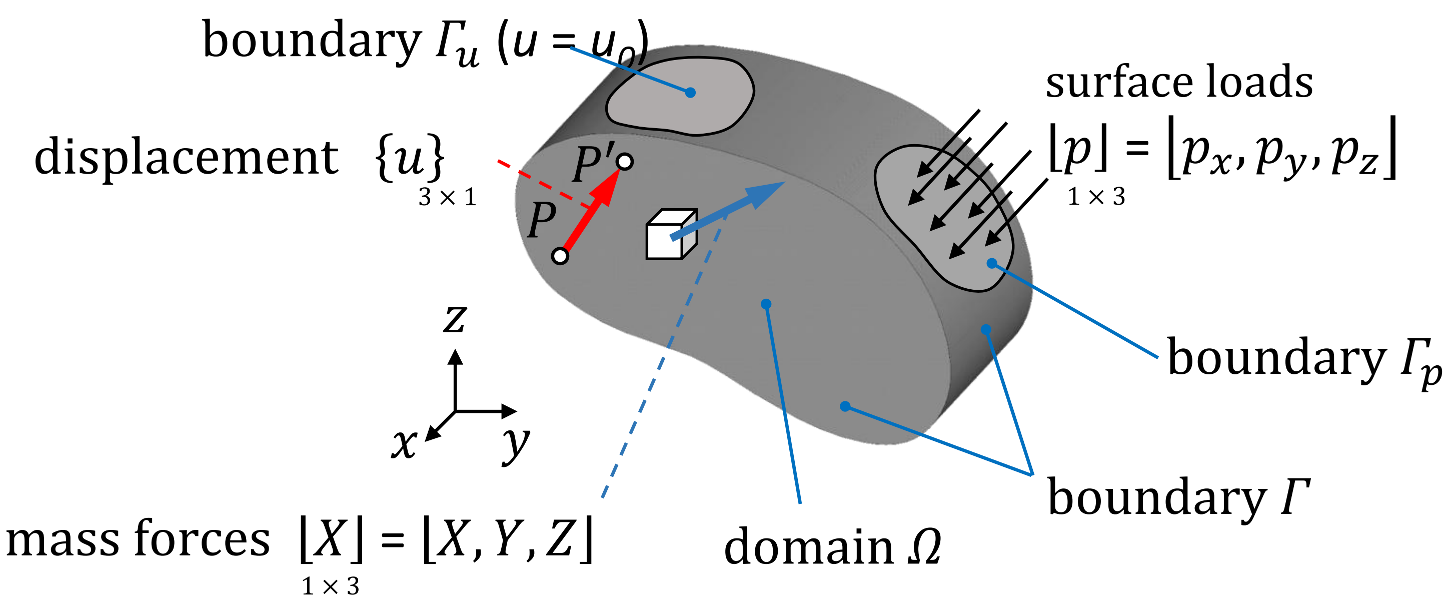
$$W = \int_{\Omega} \lfloor X \rfloor_{1\times 3} \{u\}_{3\times 1} \, d\Omega + \int_{\Gamma_p} \lfloor p \rfloor_{1\times 3} \{u\}_{3\times 1} \, d\Gamma_p$$

Minimum total potential energy principle

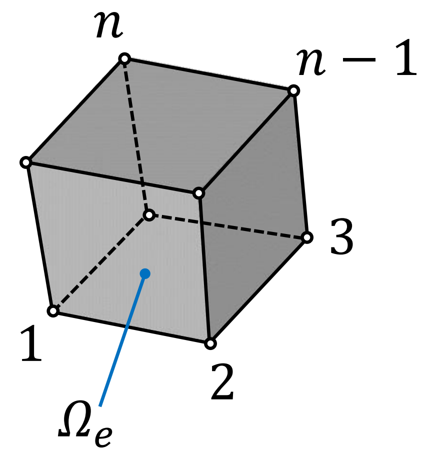
The total potential energy: $V = U - W$

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditions on Γ_u and minimizes the total potential energy V.

$$V \rightarrow min$$



Elastic strain energy in a finite element. Local stiffness matrix



$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

elastic strain energy in a finite element:

$$U_e = \frac{1}{2}\int\limits_{\Omega_e} \underset{1\times 6}{\lfloor \varepsilon \rfloor} \underset{6\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2} \underset{1\times n_e}{\lfloor q \rfloor_e} \int\limits_{\Omega_e} \underset{n_e\times 6}{[B]^T} \underset{6\times 6}{[D]} \underset{6\times n_e}{[B]}\, d\Omega_e \underset{n_e\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times n_e}{\lfloor q \rfloor_e} \underset{n_e\times n_e}{[k]_e} \underset{n_e\times 1}{\{q\}_e}$$

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]} \underset{6\times 1}{\{\varepsilon\}}$$

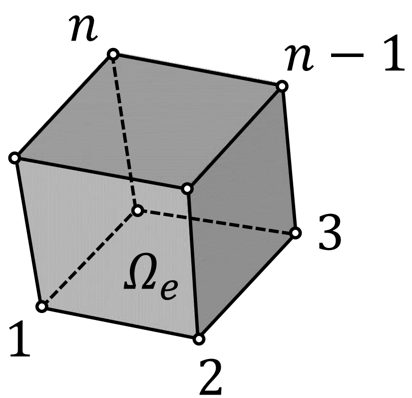
$$\underset{1\times 6}{\lfloor \varepsilon \rfloor} = \underset{1\times n_e}{\lfloor q \rfloor_e} \underset{n_e\times 6}{[B]^T} \qquad \underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]} \underset{n_e\times 1}{\{q\}_e}$$

local stiffness matrix:

$$\underset{n_e\times n_e}{[k]_e} = \int\limits_{\Omega_e} \underset{n_e\times 6}{[B]^T} \underset{6\times 6}{[D]} \underset{6\times n_e}{[B]} d\Omega_e$$

Elastic strain energy in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE:

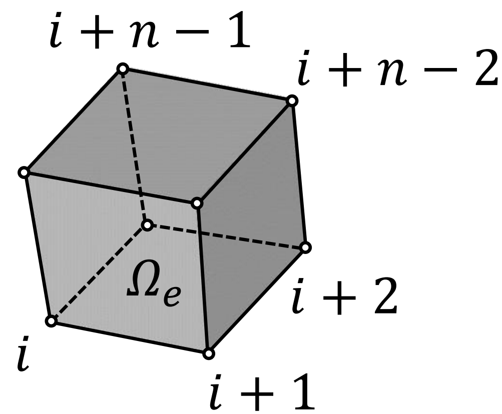
$$n_e = n \cdot n_p$$

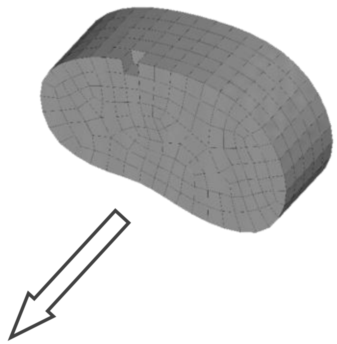
$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$U_e = \frac{1}{2} \underset{1 \times n_e}{\lfloor q \rfloor_e} \underset{n_e \times n_e}{[k]_e} \underset{n_e \times 1}{\{q\}_e}$$

↑ local stiffness matrix

global notation:





NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom :

$$NDOF = NON \cdot n_p$$

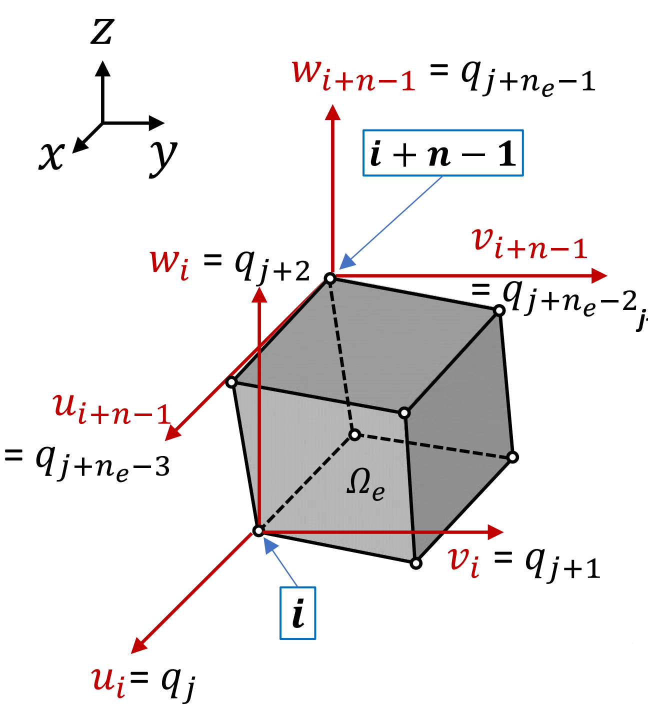
$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$U_e = \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}}$$

↑ extended local stiffness matrix

Extended local stiffness matrix of a finite element

$$\underset{NDOF\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_j \\ \vdots \\ q_{NDOF} \end{Bmatrix}$$

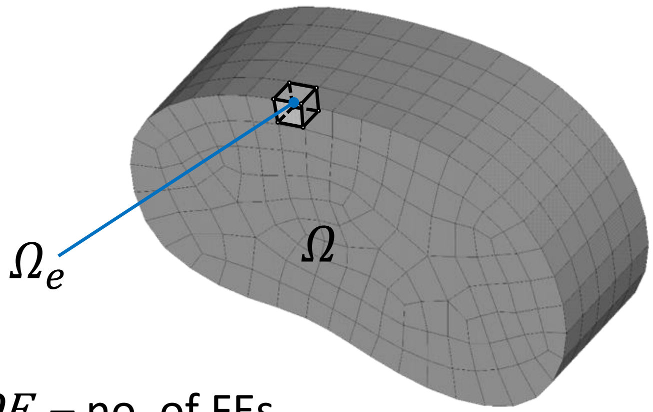


$[k]_e^* =$

	1	2	…	j−1	j	j+1	…	j+n_e−1	j+n_e	…	NDOF
1	0	0	…	0	0	0	…	0	0	…	0
2	0	0	…	0	0	0	…	0	0	…	0
…	…	…	…	0	0	0	…	0	0	…	0
j−1	0	0	0	0	0	0	…	0	0	…	0
j	0	0	0	0	k_{11}	k_{12}	…	k_{1n_e}	0	…	0
j+1	0	0	0	0	k_{21}	k_{22}	…	k_{2n_e}	0	…	0
…	…	…	…	…	…	…	…	…	0	…	0
j+n_e−1	0	0	0	0	k_{n_e1}	k_{n_e2}	…	$k_{n_en_e}$	0	…	0
j+n_e	0	0	0	0	0	0	0	0	0	…	0
…	…	…	…	…	…	…	…	…	…	…	0
NDOF	0	0	0	0	0	0	0	0	0	0	0

(assumed ascending order of components)

Elastic strain energy in a FE model. Global stiffness matrix



NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \quad \rightarrow \quad U = \sum_{e=1}^{NOE} U_e$$

$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

elastic strain energy in a finite element model:

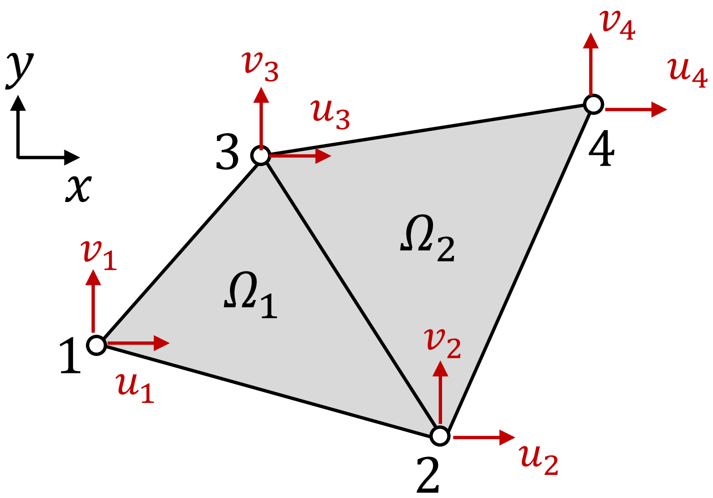
$$U = \sum_{e=1}^{NOE} U_e = \sum_{e=1}^{NOE} \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} = \frac{1}{2} \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \sum_{e=1}^{NOE} \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} =$$

$$= \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}}$$

global stiffness matrix:

$$\underset{NDOF \times NDOF}{[K]} = \sum_{e=1}^{NOE} [k]_e^*$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles

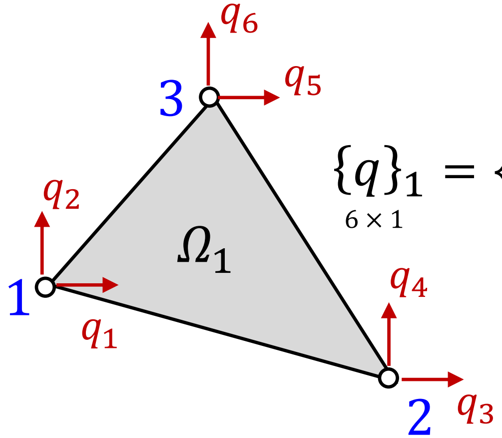


global notation:

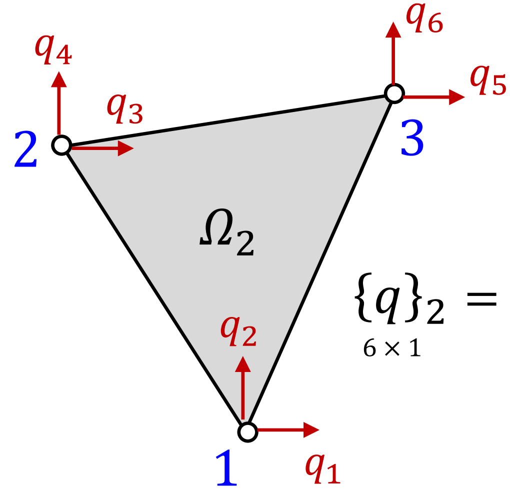
$NOE = 2$
$NON = 4$
$n = 3$
$n_p = 2 \quad ; \quad (u, v)$
$n_e = n \cdot n_p = 6$
$NDOF = NON \cdot n_p = 8$

$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

local notation:



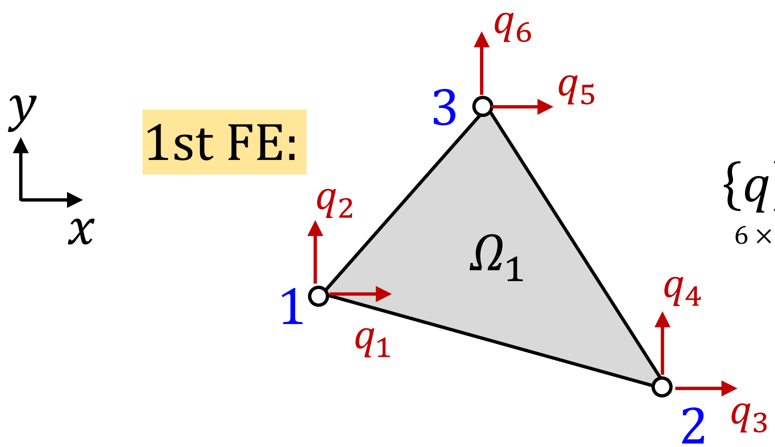
$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1$$



$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles

1st FE:



$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1 \Rightarrow \underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$\underset{6\times 6}{[k]_1} =$

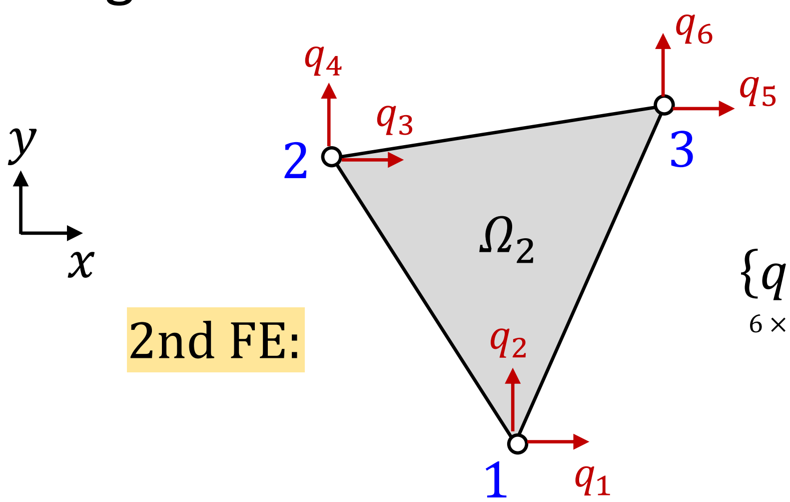
	1	2	3	4	5	6
1	a_1	b_1	c_1	d_1	e_1	f_1
2	b_1	g_1	h_1	i_1	j_1	k_1
3	c_1	h_1	l_1	m_1	n_1	o_1
4	d_1	i_1	m_1	p_1	r_1	s_1
5	e_1	j_1	n_1	r_1	t_1	$\bar{u}_1$
6	f_1	k_1	o_1	s_1	$\bar{u}_1$	$\bar{w}_1$

$\underset{8\times 8}{[k]_1^*} =$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1	m_1	n_1	o_1	0	0
4	d_1	i_1	m_1	p_1	r_1	s_1	0	0
5	e_1	j_1	n_1	r_1	t_1	$\bar{u}_1$	0	0
6	f_1	k_1	o_1	s_1	$\bar{u}_1$	$\bar{w}_1$	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0

Example 4: global stiffness matrix of a 2D model with two 3-node triangles

2nd FE:



$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2 \Rightarrow \underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

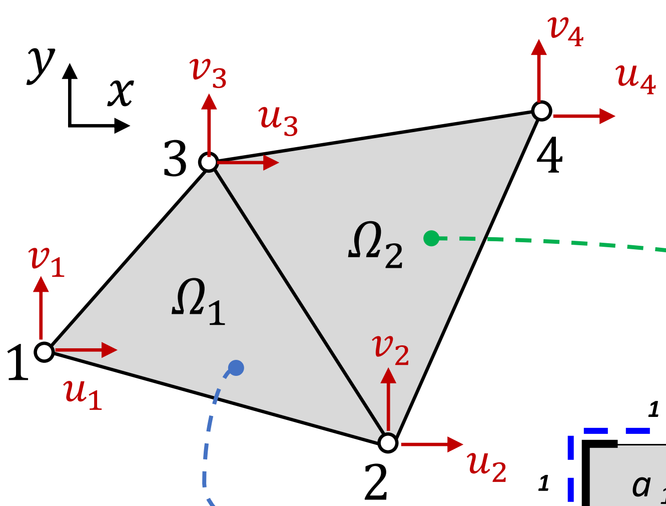
$\underset{6\times 6}{[k]_2} =$

	1	2	3	4	5	6
1	a_2	b_2	c_2	d_2	e_2	f_2
2	b_2	g_2	h_2	i_2	j_2	k_2
3	c_2	h_2	l_2	m_2	n_2	o_2
4	d_2	i_2	m_2	p_2	r_2	s_2
5	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
6	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$\underset{8\times 8}{[k]_2^*} =$

	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	a_2	b_2	c_2	d_2	e_2	f_2
4	0	0	b_2	g_2	h_2	i_2	j_2	k_2
5	0	0	c_2	h_2	l_2	m_2	n_2	o_2
6	0	0	d_2	i_2	m_2	p_2	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles



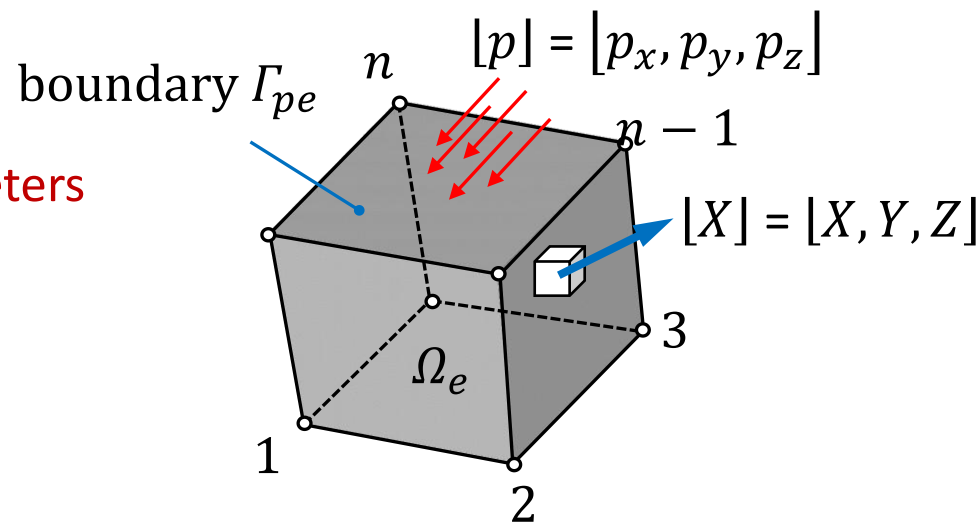
$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underset{8\times 8}{[K]} = \underset{8\times 8}{[k]_1^*} + \underset{8\times 8}{[k]_2^*} =$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

Potential energy of loading in a finite element

$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters



potential energy of loading
in a finite element :

$$W_e = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times 1}{\{u\}}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times 1}{\{u\}}\, d\Gamma_{pe} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}\, d\Gamma_{pe} =$$

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N]} \underset{n_e\times 1}{\{q\}_e}$$

$$= \left(\int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]}\, d\Gamma_{pe}\right) \underset{n_e\times 1}{\{q\}_e} = \left(\underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}\right) \underset{n_e\times 1}{\{q\}_e} = \underset{1\times n_e}{\lfloor F \rfloor_e} \underset{n_e\times 1}{\{q\}_e}$$

equivalent load vector:

$$\underset{1\times n_e}{\lfloor F \rfloor_e} = \underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}$$

Equivalent load vector

$$\underset{1\times\, n_e}{\lfloor F \rfloor_e} = \underset{1\times\, n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}$$

equivalent load vector due to mass forces:

$$\underset{1\times\, n_e}{\lfloor F^X \rfloor_e} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]} d\Omega_e =$$

$$= \int_{\Omega_e} \lfloor X, Y, Z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \;\ldots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_n \end{bmatrix} d\Omega_e$$

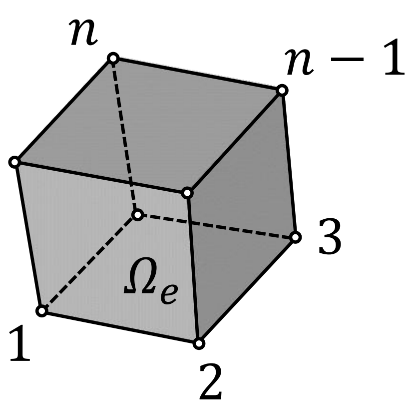
equivalent load vector due to surface load:

$$\underset{1\times\, n_e}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]} d\Gamma_{pe} =$$

$$= \int_{\Gamma_{pe}} \lfloor p_x, p_y, p_z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \;\ldots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_n \end{bmatrix} d\Gamma_{pe}$$

Potential energy of loading in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE :

$$n_e = n \cdot n_p$$

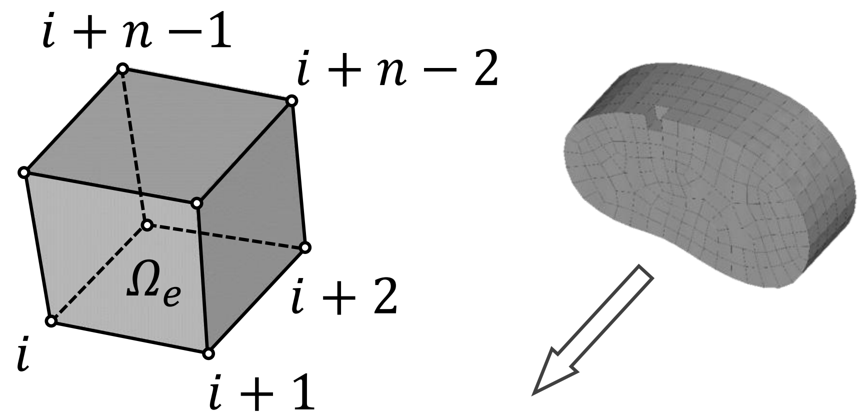
$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$W_e = \underset{1 \times n_e}{\lfloor q \rfloor_e} \underset{n_e \times 1}{\{F\}_e}$$

↑

equivalent load vector

global notation:



NON – no. of nodes

n_p – no. of nodal parameters per no

no. of degrees of freedom :

$$NDOF = NON \cdot n_p$$

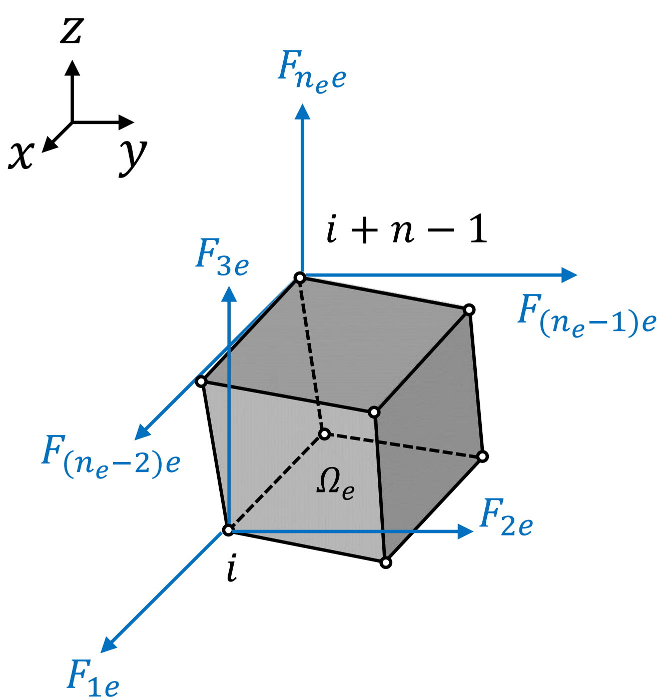
$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$W_e = \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times 1}{\{F\}_e^*}$$

↑

extended equivalent load vector

Extended equivalent load vector in a finite element



extended equivalent load vector:

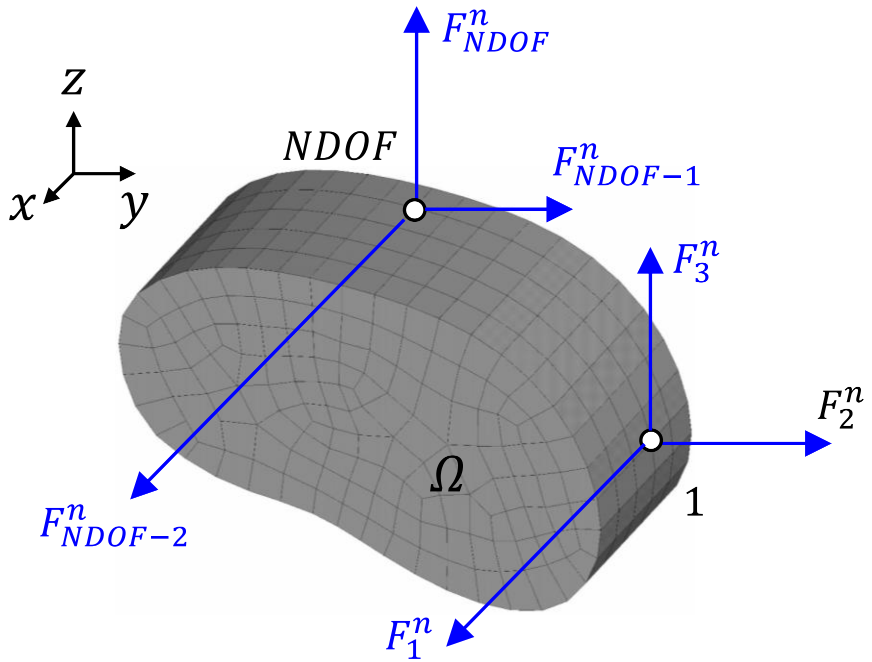
$$\underset{NDOF\times 1}{\{F\}_e^*} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ F_{1e} \\ F_{2e} \\ \dots \\ F_{n_e e} \\ 0 \\ \dots \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ \\ j-1 \\ j \\ j+1 \\ \\ j+n_e-1 \\ j+n_e \\ \\ NDOF \end{matrix}$$

$(assumed\ ascending\ order\ of\ components)$

equivalent load vector:

$$\underset{n_e\times 1}{\{F\}_e} = \begin{Bmatrix} F_{1e} \\ F_{2e} \\ F_{3e} \\ \dots \\ F_{(n_e-2)e} \\ F_{(n_e-1)e} \\ F_{n_e e} \end{Bmatrix}$$

Forces applied directly at nodes. Potential energy of nodal loads



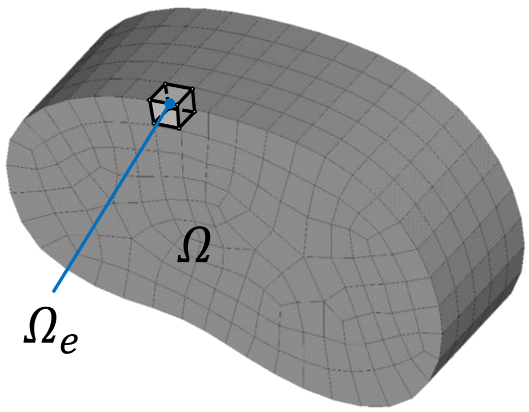
nodal load vector:

$$\underset{NDOF\times 1}{\{F\}^n} = \begin{Bmatrix} F^n_1 \\ F^n_2 \\ F^n_3 \\ \dots \\ F^n_{NDOF-2} \\ F^n_{NDOF-1} \\ F^n_{NDOF} \end{Bmatrix}$$

potential energy of nodal loads:

$$W^n = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n}$$

Potential energy of loading in a FE model. Global load vector



NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

potential energy of element loads:

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \rightarrow W^e = \sum_{e=1}^{NOE} W_e$$

potential energy of nodal loads

potential energy of loading in a finite element model: $W = W^e + W^n$

$$W = \sum_{e=1}^{NOE} W_e + W^n = \sum_{e=1}^{NOE} \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}_e^*} + \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n} = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \Big(\sum_{e=1}^{NOE} \underset{NDOF\times 1}{\{F\}_e^*} + \underset{NDOF\times 1}{\{F\}^n}\Big)$$

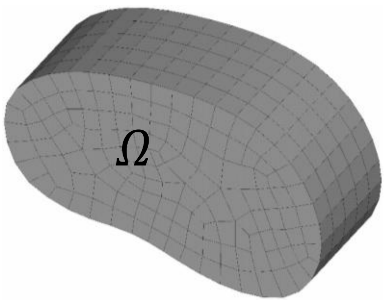
$$= \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \big(\underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}\big) \rightarrow W = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}}$$

global load vector of element loads

nodal load vector

global load vector: $\underset{NDOF\times 1}{\{F\}} = \underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}$

Total potential energy in a FE model. Set of linear equations



NOE – no. of FEs
$NDOF$ – no. of degrees of freedom

Total potential energy of the entire model:

$$V = U - W = \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} - \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times 1}{\{F\}}$$

$$\underset{NDOF \times 1}{\{q\}} = ?$$

$$V \rightarrow min$$

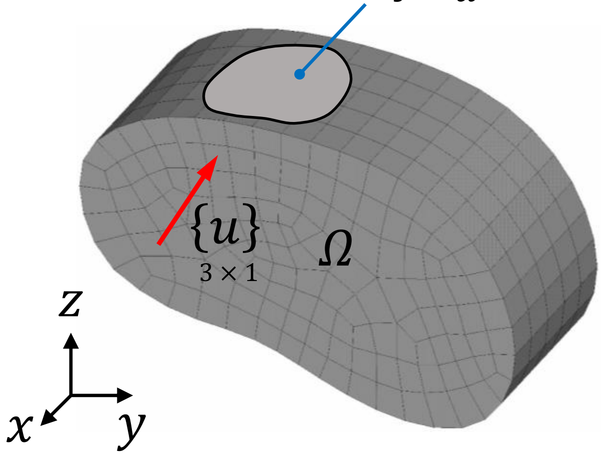
$$\frac{\partial V}{\partial q_j} = 0 \rightarrow \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} = \underset{NDOF \times 1}{\{F\}}$$

set of linear algebraic equations

$$\det \left(\underset{NDOF \times NDOF}{[K]} \right) = 0$$

Set of FE equations with boundary conditions

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditons on Γ_u and minimizes the total potential energy *V*.



NDOF – no. of degrees of freedom

NOF – no. of known degrees of freedom on Γ_u

N – number of unknown degrees of freedom:

$$N = NDOF - NOF$$

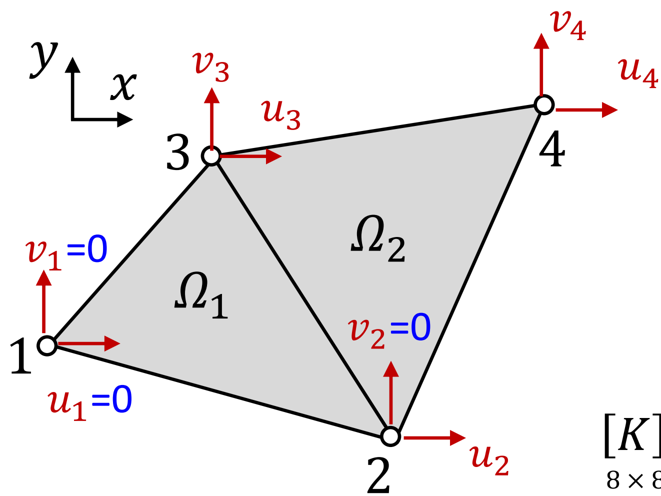
$$\underset{NDOF\times NDOF}{[K]} \rightarrow \underset{N\times N}{[K]} \;;\; \underset{NDOF\times 1}{\{q\}} \rightarrow \underset{N\times 1}{\{q\}} \;;\; \underset{NDOF\times 1}{\{F\}} \rightarrow \underset{N\times 1}{\{F\}}$$

$$\underset{N\times N}{[K]} \cdot \underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}}$$

$$\det\left(\underset{N\times N}{[K]}\right) \neq 0$$

linear set of algebraic equations with boundary conditions

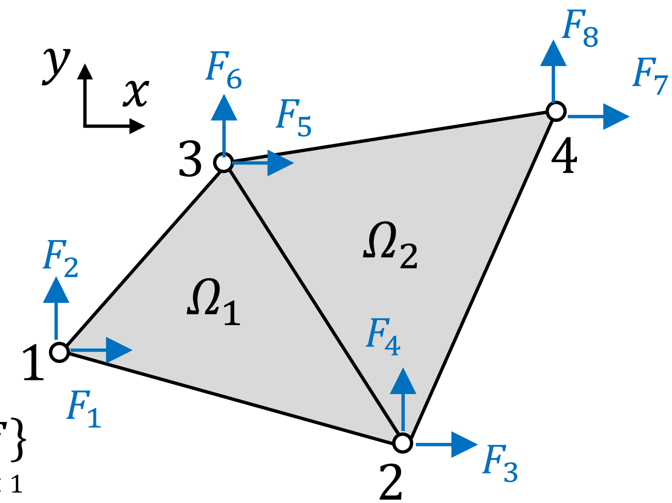
Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles



$NDOF = 8$

$NOF = 3$

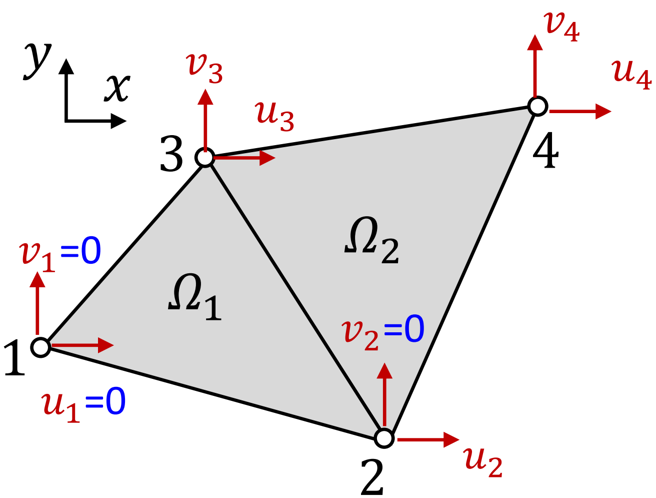
$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$



	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles

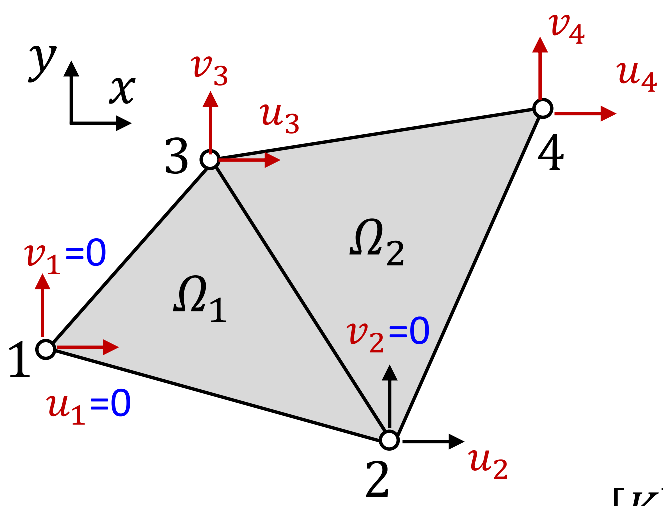


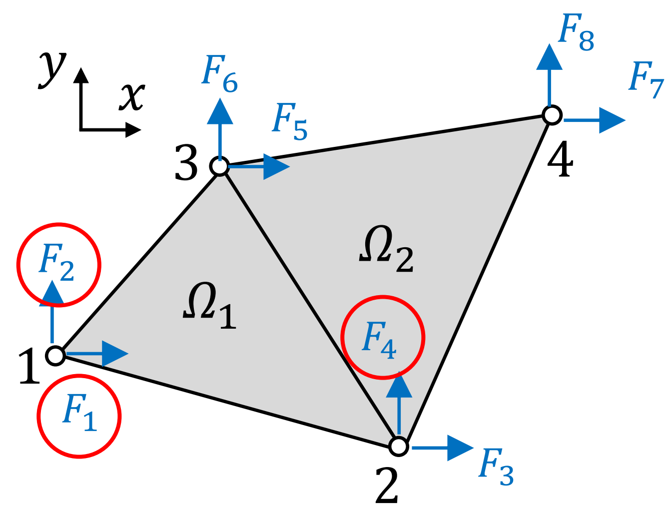
$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles





$N = 8 - 3 = 5$

$$\underset{5\times 5}{[K]} \cdot \underset{5\times 1}{\{q\}} = \underset{5\times 1}{\{F\}}$$

l_1+a_2	n_1+c_2	o_1+d_2	e_2	f_2
n_1+c_2	t_1+l_2	u_1+m_2	n_2	o_2
o_1+d_2	u_1+m_2	w_1+p_2	r_2	s_2
e_2	n_2	r_2	t_2	$\bar{u}_2$
f_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

linear set of algebraic equations with boundary conditions

Solution of a set of FE equations with boundary conditions

$$\underset{N\times N}{[K]}\cdot\underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}} \;\rightarrow\; \det\left(\underset{N\times N}{[K]}\right)\neq 0 \;\rightarrow\; \underset{N\times 1}{\{q\}} = \underset{N\times N}{[K]^{-1}}\underset{N\times 1}{\{F\}}$$

DOF solution: $\underset{NDOF\times 1}{\{q\}}$

Element solution (ES): $\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$; $\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 6}{[D]}\underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$

↑ strain in a finite element ↑ stress in a finite element

Nodal solution (NS):

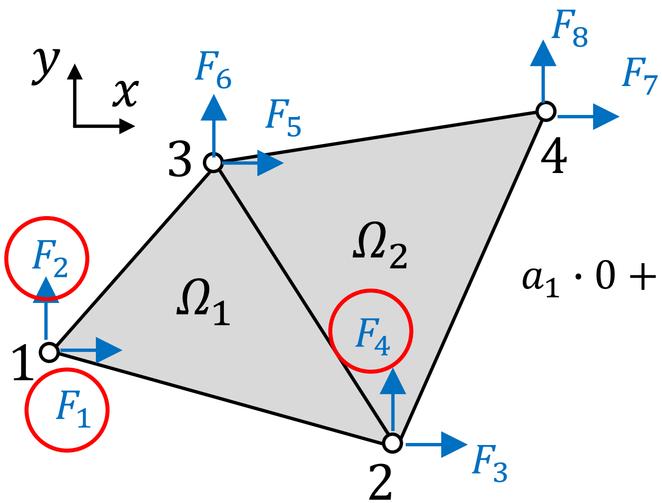
$$(NS)_i = \frac{\sum_{e=1}^{k}(ES)_{ei}}{k}$$

$(NS)_i$ – avaraged nodal solution at node (i)

$(ES)_{ei}$ – element solution in element (e) and at node (i)

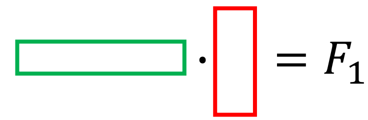
k – no. of elements adjacent to node (i)

Example 6. Reactions calculation for 2D problem. FE model with two 3-node triangles

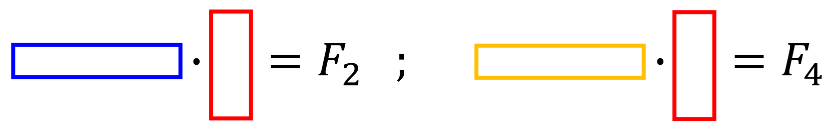


$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

$\{q\}$ ← *known*



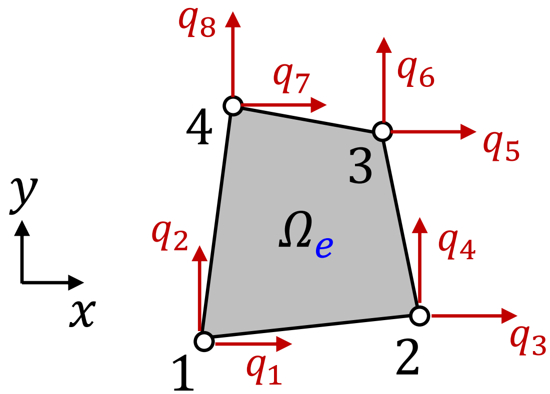
$$a_1 \cdot 0 + b_1 \cdot 0 + c_1 \cdot u_2 + d_1 \cdot 0 + e_1 \cdot u_3 + f_1 \cdot v_3 + 0 \cdot u_4 + 0 \cdot v_4 = F_1$$



	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

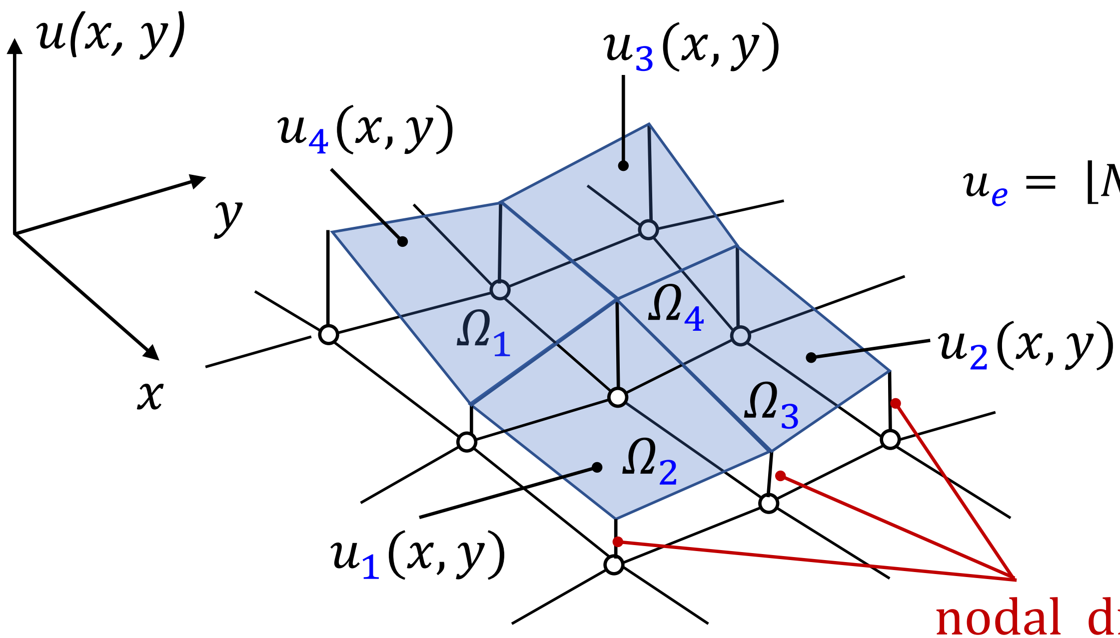
$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 7. DOF solution $u(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



$$\underset{2\times 1}{\{u\}} = \underset{2\times 8}{[N]}\underset{8\times 1}{\{q\}_e}$$

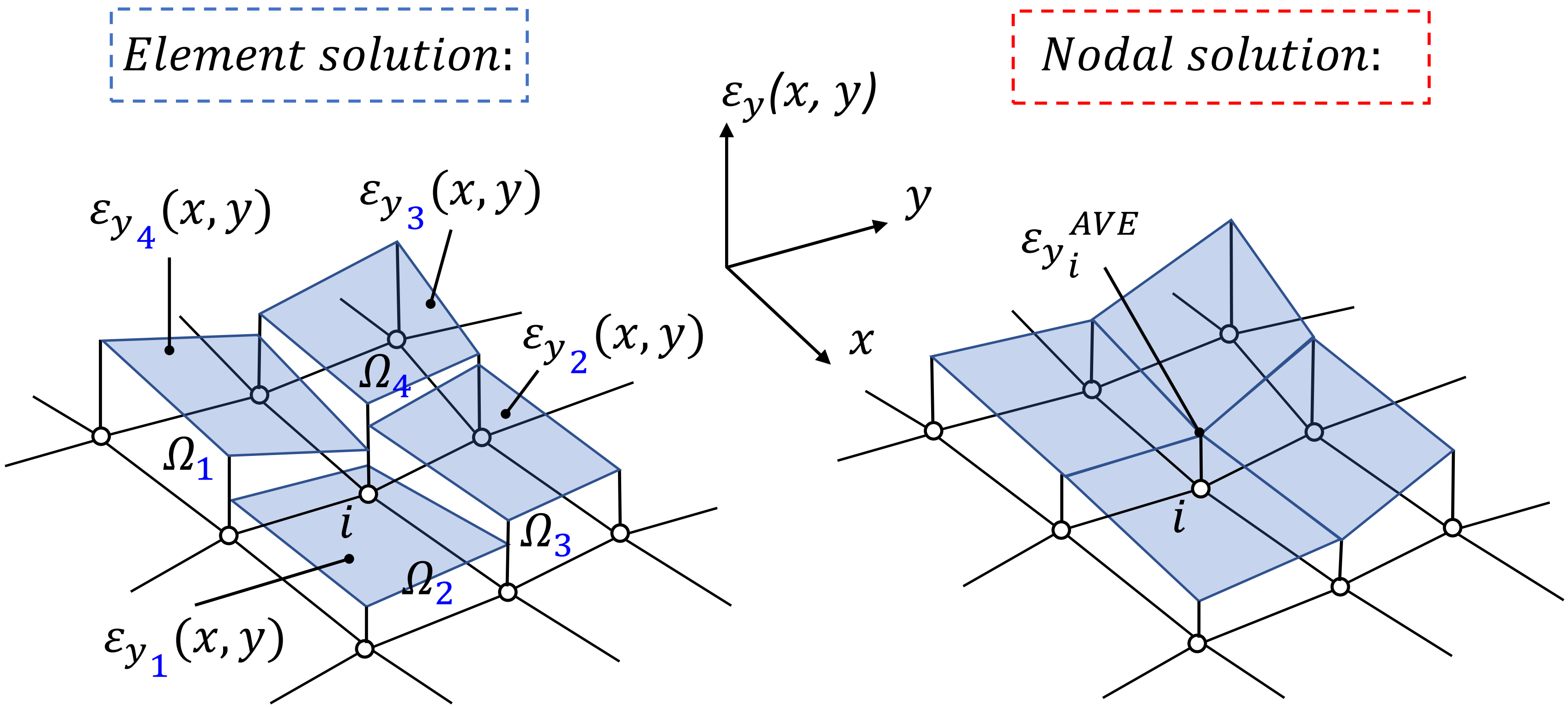
$u_e(x,y)$ – displacement in x direction



$$u_e = \lfloor N_1\ 0\ \ N_2\ 0\ \ N_3\ 0\ \ N_4\ 0 \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}_e$$

nodal displacement in x direction

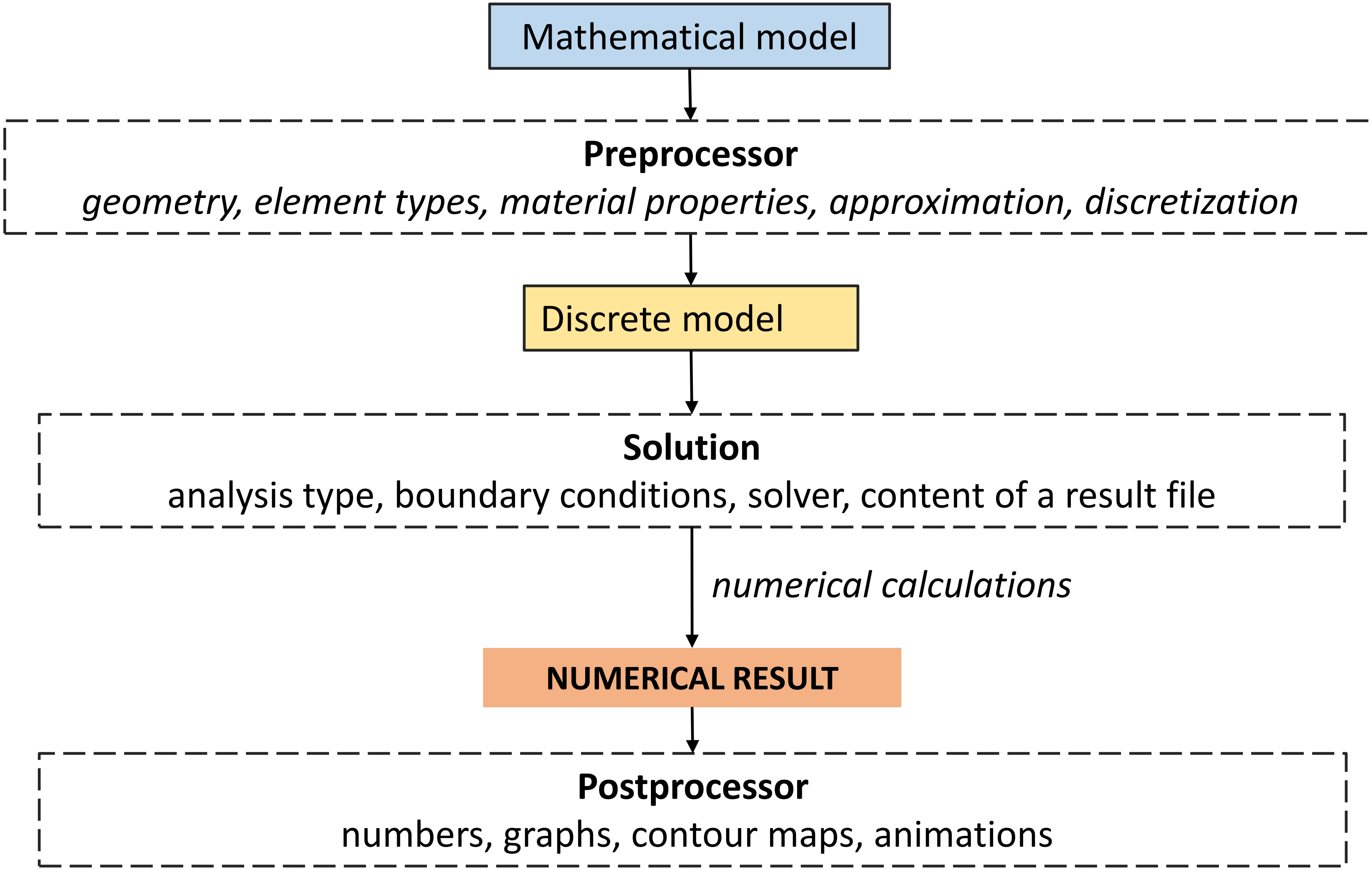
Example 8. Strain component $\varepsilon_y(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



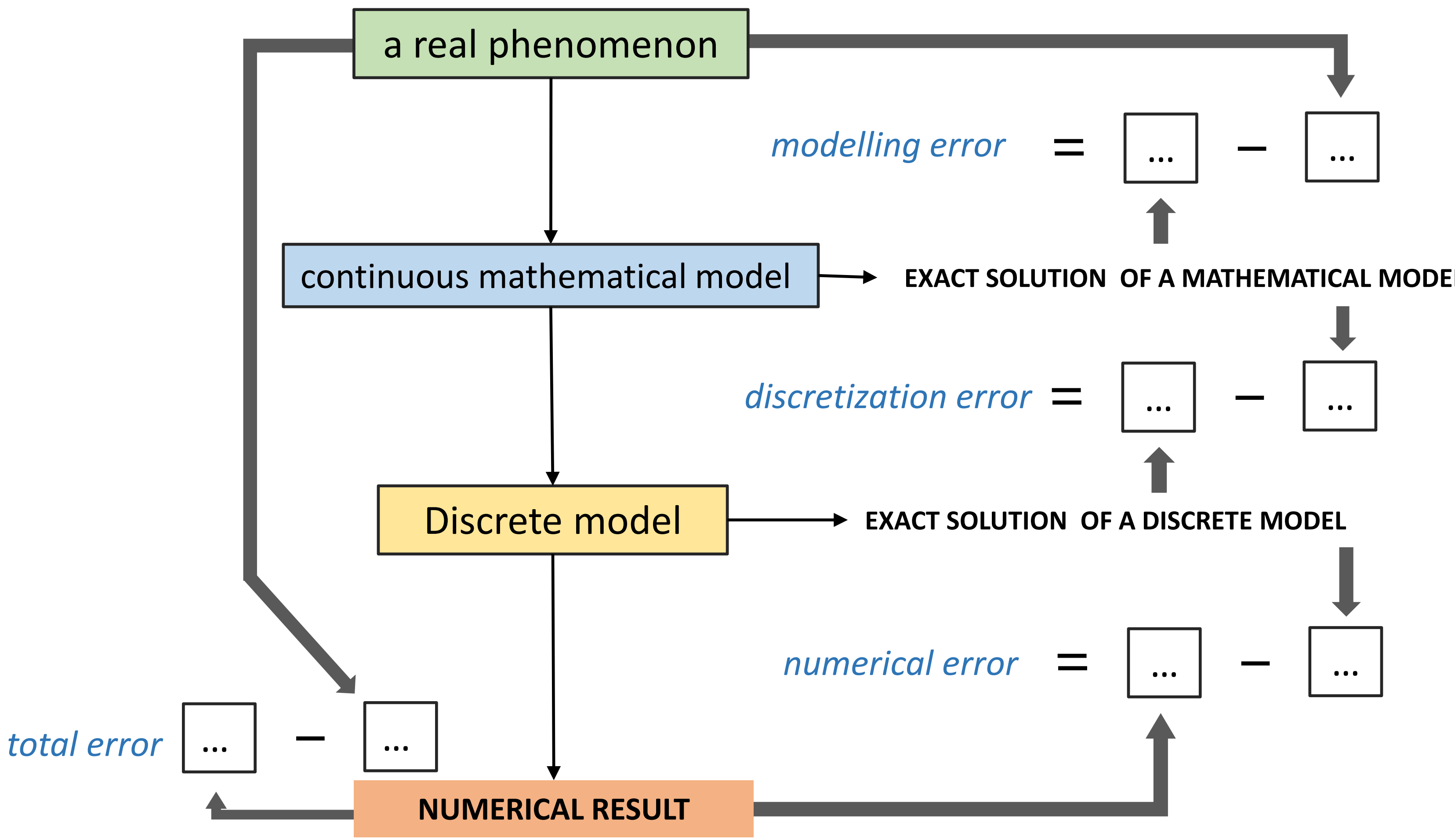
$k = 4$

$$\varepsilon_{y_i}^{AVE} = \frac{\varepsilon_{y_1}(x_i, y_i) + \varepsilon_{y_2}(x_i, y_i) + \varepsilon_{y_3}(x_i, y_i) + \varepsilon_{y_4}(x_i, y_i)}{4}$$

FE modelling – basic steps



Accuracy of FEM calculations



total error = modelling error + discretization error + numerical error

modelling error ≈ discretization error ≈ numerical error → min